·Lecture #24·

July 14, 2015

** Finished Example from previous lecture! **

Example:

$$\begin{cases} x' = (1-x-y)x \\ y' = (0.5 - 0.25y - 0.75x)y \end{cases}$$

Critical Points: $(0,0)$, $(0,2)$, $(1,0)$, $(0.5, 0.5)$

$$J(x,y) = \begin{bmatrix} 1-2x-y & -x \\ -0.25y & 0.5-0.5y-0.75x \end{bmatrix}$$

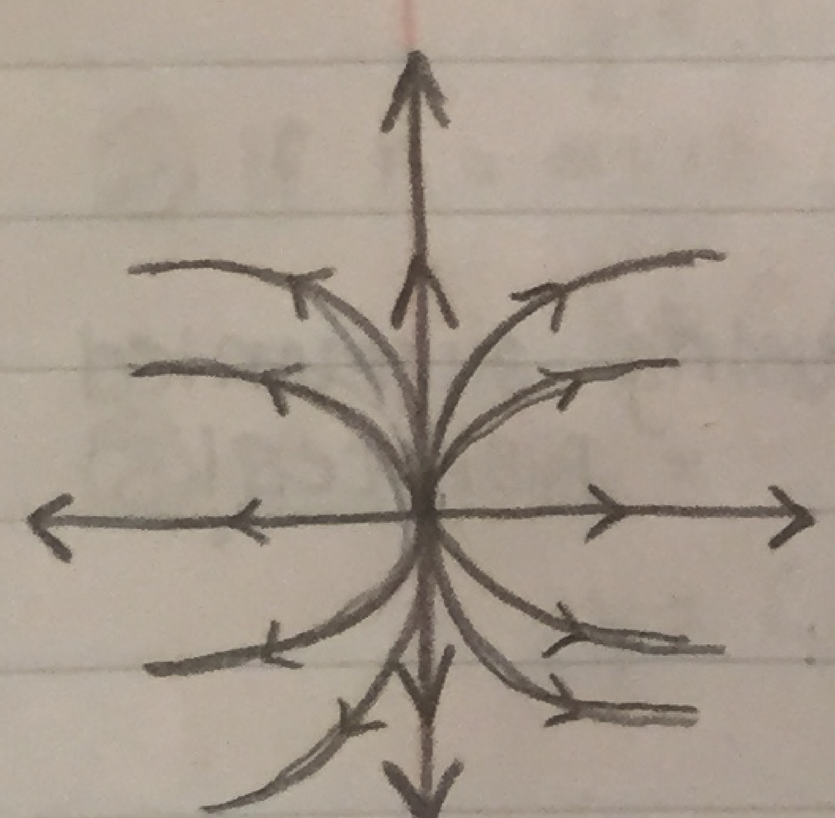
· Near $(0,0)$: $\vec{u}' = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \vec{u}$

Eigenvalues:

$\lambda_1 = 1$, $\lambda_2 = 0.5$

Eigenvectors:

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$\Rightarrow$ $(0,0)$ is Nodal Source (unstable)

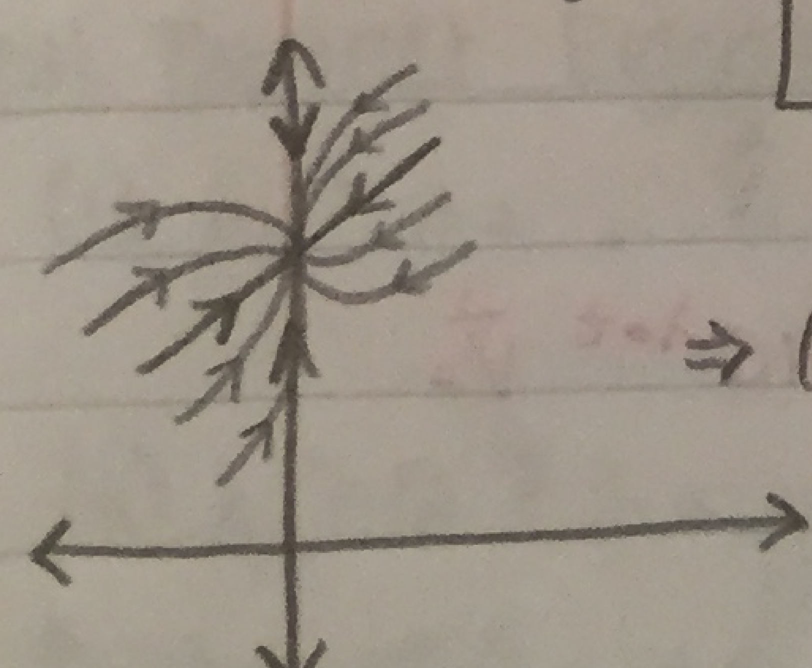
· Near $(0,2)$: $\vec{u}' = \begin{bmatrix} -1 & 0 \\ -1.5 & -0.5 \end{bmatrix} \vec{u}$

Eigenvalues:

$\lambda_1 = -1$, $\lambda_2 = -0.5$

Eigenvectors:

$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$\Rightarrow$ $(0,2)$ is a Nodal Sink (stable)

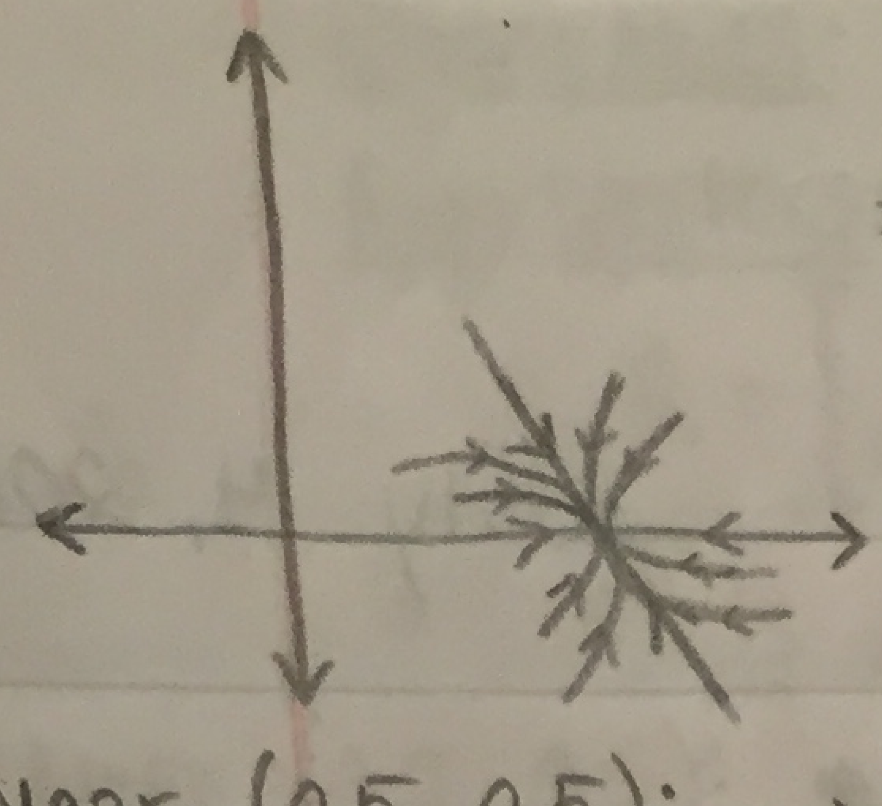
· Near $(1,0)$: $\vec{u}' = \begin{bmatrix} -1 & -1 \\ 0 & -0.25 \end{bmatrix} \vec{u}$

Eigenvalues:

$\lambda_1 = -1$, $\lambda_2 = -0.25$

Eigenvectors:

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ $\longrightarrow$



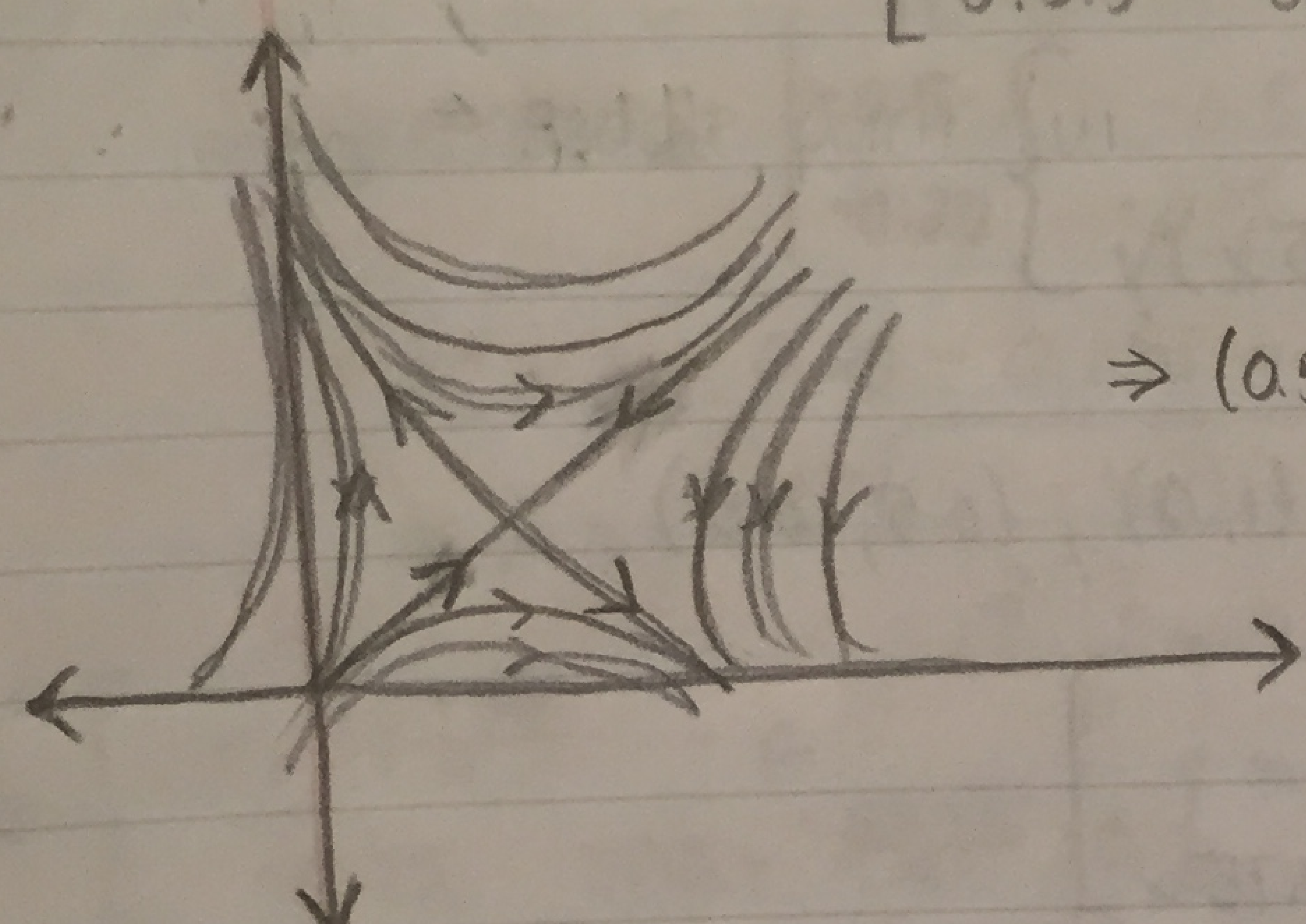
$\Rightarrow$ (1,0) is a Nodal Sinh (Stable)

• Near (0.5, 0.5): $\vec{u}' = \begin{bmatrix} -0.5 & -0.5 \\ -0.375 & -0.125 \end{bmatrix} \vec{u}$

Eigenvalues:

$\lambda_1 = -\frac{5}{16} + \frac{\sqrt{28}}{16} > 0$

$\lambda_2 = = -\frac{5}{16} - \frac{\sqrt{29}}{16} < 0$



$\Rightarrow$ (0.5,.0.5) is a Saddle Point (unstable)

$\Rightarrow$ One of the species is to exsist, depending on initial conditions!

FINAL EXAM REVIEW

Part 3: Systems of ODEs

• How to solve $A\vec{x} = \vec{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\rightarrow$ using augmented matrices!

• How to find Eigenvalues & Eigenvectors

• How to Solve 2x2 Systems, $\vec{x}' = A\vec{x}$

① Real Distinct Eigenvalues:

$\lambda_1 \neq \lambda_2$, both are REAL!

$\Rightarrow \vec{x}(t) = C_1 e^{\lambda_1 t} \vec{V}_1 + C_2 e^{\lambda_2 t} \vec{V}_2$

② Complex Eigenvalues:

$\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$

Let $\vec{V}_1 = \vec{a} + i\vec{b}$

$\Rightarrow \vec{x}(t) = C_1 e^{\alpha t}(\cos\beta t\,\vec{a} - \sin\beta t\,\vec{a}) + C_2 e^{\alpha t}(\sin\beta t\,\vec{a} + \cos\beta t\,\vec{a})$

• Alternatively, seperate the real + imaginary parts! $\longrightarrow$

$\Rightarrow e^{(\alpha+i\beta)t}\,\vec{v}_1 = \vec{u} + i\vec{w}$

$\Rightarrow$ General Solution:

$$\vec{x}(t) = C_1\vec{u}(t) + C_2\vec{v}(t)$$

3 Real, Repeated Roots:

$$\lambda_1 = \lambda_2 = \lambda$$

$\rightarrow$ If only "one" eigenvector can be found, let $\vec{v}$ be the eigenvector, $\vec{w}$ will be the generalized eigenvector

$$((A-\lambda I)\vec{w} = \vec{v})$$

General Solution:

$$\vec{x}(t) = C_1 e^{\lambda t}\vec{v} + C_2 e^{\lambda t}(t\vec{v} + \vec{w})$$

*Recall: $ay'' + by' + cy = 0$

r_1, r_2 are characteristic roots, i.e., $ar^2 + br + c = 0$

1 If $r_1 \neq r_2$, real:

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

2 If $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$:

$$y = C_1 e^{\alpha t}\cos\beta t + C_2 e^{\alpha t}\sin\beta t$$

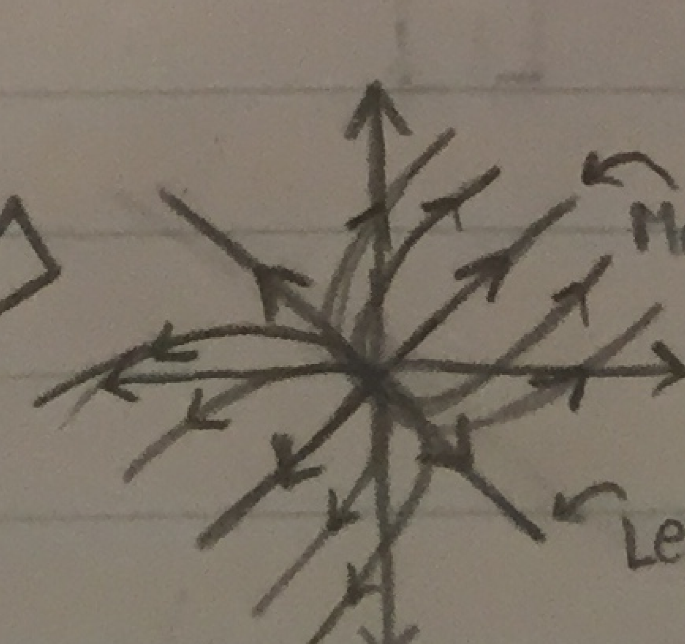
3 If $r_1 = r_2 = r$:

$$y = C_1 e^{rt} + C_2 t e^{rt}$$

- How to draw phase portrait?

Real Distinct Eigenvalues:

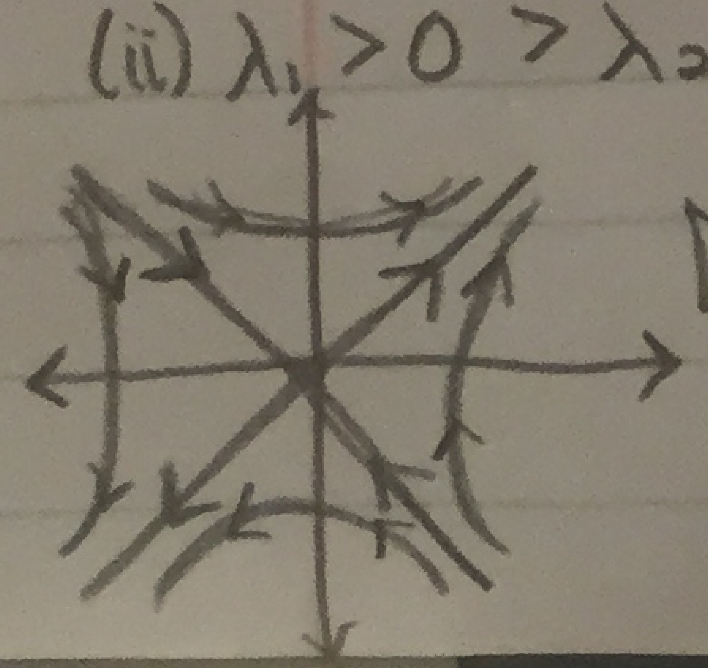
(i) $\lambda_1 > \lambda_2$, $\vec{v}_1$, $\vec{v}_2$ $\Rightarrow$



More positive Eigenvalue

Less positive Eigenvalue

(ii) $\lambda_1 > 0 > \lambda_2$



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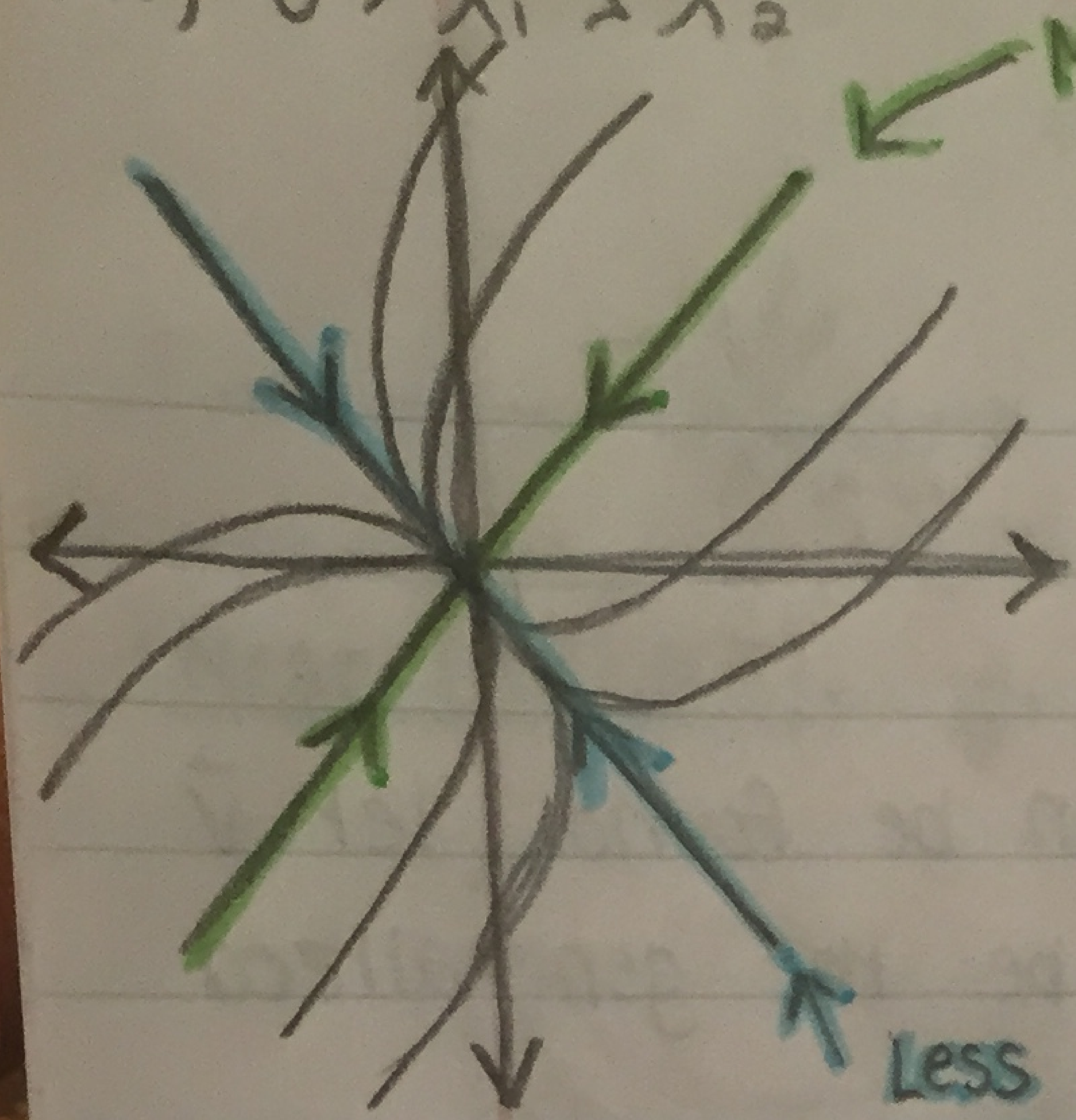
(iii) $0 > \lambda_1 > \lambda_2$



* **Note**: The Body of the butterfly will lie along the eigenvalue closer to zero, where the wings will open up to the Larger eigenvalue (in this case, body is less negative eigenvalue and wings open up to more negative eigenvalue)!

Complex Eigenvalues:

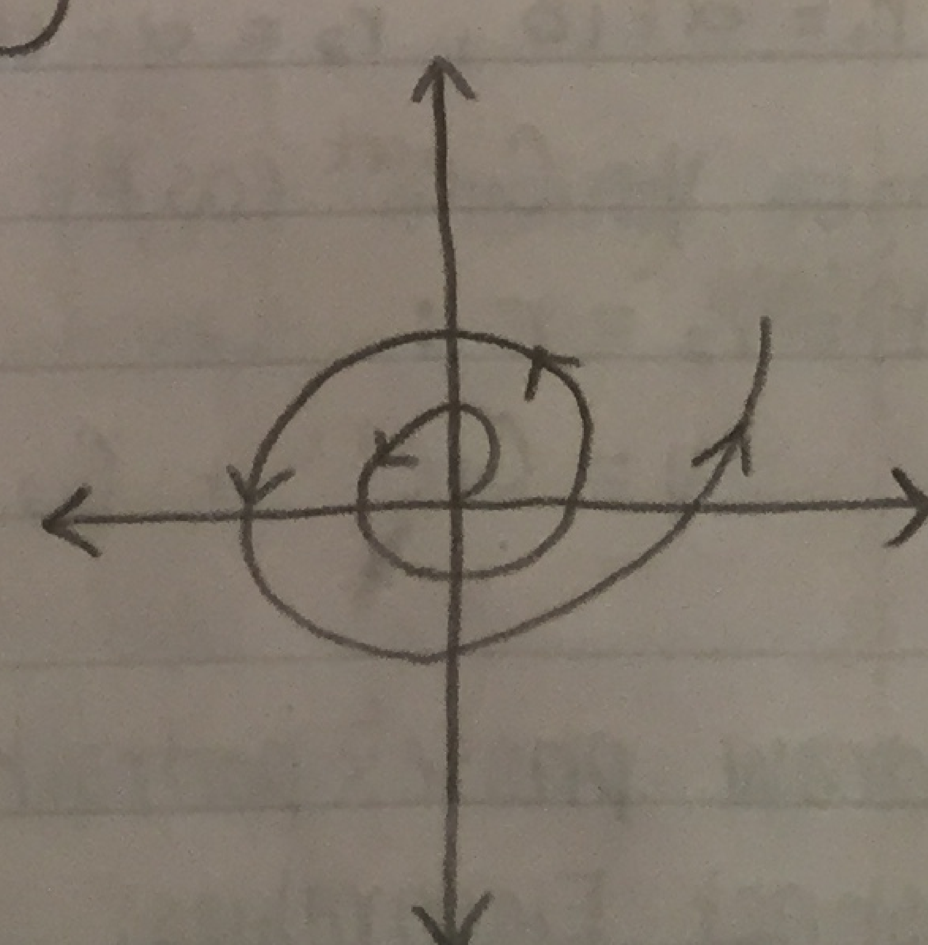
$\lambda = \alpha \pm i\beta$

① Use α to decide the type
② Use the tangent vector at (1,0) to determine orientation

$$\begin{cases} \alpha > 0 & \text{spiral source} \\ \alpha = 0 & \text{center} \\ \alpha < 0 & \text{spiral sink} \end{cases}$$

Example: $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \vec{x}$

$\lambda = 2 \pm i$ $\quad \text{Re}\,\lambda > 0 \Rightarrow$ Spiral Source



Tangent vector at (1,0) is

$\vec{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$